POTENTIAL THEORY AND ASYMPTOTICS FOR ORTHOGONAL POLYNOMIALS

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In this introductory talk we are going to see that potential theoretic methods are very useful for establishing asymptotics of orthogonal polynomials. After we introduce the definition of orthogonal polynomials and recall some important tools from potential theory (Chebyshev polynomials, equilibrium measures and the principle of descent), our aim is to prove the following theorem concerning the zero distribution of orthogonal polynomials.

Theorem. Let $d\mu(x) = w(x)dx$ be a finite Borel measure supported on $[-1, 1]$, let $p_n(x) = \gamma_n x^n + \dots$, $\gamma_n > 0$ be the orthonormal polynomial of degree n and let $x_{1n}, \dots, x_{nn}$ denote its zeros. Define the normalized counting measure of the zeros as

$$\mu_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{x_{kn}},$$

where δ_a is the Dirac probability measure concentrated on $\{a\}$. If $w(x) > 0$ a.e. in $[-1, 1]$, then
(i) ${\gamma_n}^{-1/n} \to \frac{1}{2}$, where γ_n is the leading coefficient of the orthonormal polynomial p_n,
(ii) $\mu_n \xrightarrow{w} \nu$, where "$\xrightarrow{w}$" denotes weak convergence of measures and ν denotes the equilibrium measure of $[-1, 1]$, i.e. $d\nu(x) = \frac{1}{\pi\sqrt{1-x^2}}dx$.

□

What is remarkable in this theorem is the fact that under a mild condition, the zero distribution of the orthogonal polynomials always converges to the same measure regardless of the starting measure μ. If time allows, we shall discuss some possible generalizations of this theorem.

References

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