Orthogonal polynomials and operator theory

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The goal of this talk is to show the connections of spectral theory of operators with orthogonal polynomials on the real line and introduce the main ideas. In the first chapter we recall some facts from the spectral theory and we put them into use in the second chapter.

1 The spectral theorem

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H} | T \text{ is linear}\}$ denote the set of linear operators on $\mathcal{H}$ and also let $\mathcal{B}(\mathcal{H}) := \{T \in \mathcal{L}(\mathcal{H}) : T \text{ is bounded}\}$ denote the set of bounded linear operators. $\sigma(T)$ always denotes the spectrum of the operator T, A always denotes a self-adjoint operator (i.e. $A^* = A$) and $N \in \mathcal{L}(\mathcal{T})$ always denotes a normal operator (i.e. $NN^* = N^*N$, or in other words, it commutes with its adjoint). This section is devoted to the study of the spectral theorem.

1.1 Functional calculus

Suppose that N is a bounded normal operator. Before we state the spectral theorem, we shall need to interpret $f(N)$, where $f : \sigma(N) \to \mathbb{C}$ is a continuous function defined on the spectrum of N. First, let

$$p(z) = \sum_{l,k=0}^{n} c_{lk} z^l \overline{z}^k.$$

Then we have no problem defining $p(N)$ with

$$p(N) = \sum_{l,k=0}^{n} c_{lk} N^l N^{*k}.$$

Note that $NN^* = N^*N$ is needed, since $p(N)$ is not well-defined otherwise. Using the Stone-Weierstrass theorem, one can define $f(N)$ for arbitrary $f \in C(\sigma(N))$ as a limit of operators of the form $p(N)$ in the norm topology. Overall, we have the following theorem.

Theorem 1.1. *If $N \in \mathcal{B}(\mathcal{H})$ is a normal operator, then the previously defined map $C(\sigma(N)) \ni f \to f(N) \in \mathcal{B}(\mathcal{H})$ is a well-defined isometric C^*-algebra homeomorphism of $C(X)$ onto the C^*-algebra generated by N and I.*

1.2 Multiplication operators

Definition 1.2. Let $\mathcal{H} = L^2(X, \mu)$ be a given Hilbert space and let $f \in L^\infty(X, \mu)$. The operator $M_f : L^2(X, \mu) \to L^2(X, \mu)$ defined with

$$(M_f g)(x) := f(x)g(x)$$

is called a multiplication operator.

There is a lot known about multiplication operators, for example $\|f\|_\infty = \|M_f\|$ and

$$\sigma(M_f) = \{x \in X : \mu(|f - x| < \varepsilon) > 0 \text{ for all } \varepsilon > 0\}.$$

The latter set is called the essential range of f. Using multiplication operators, the spectral theorem states the following.

Theorem 1.3 (Spectral theorem)**.** *For every $N \in \mathcal{B}(\mathcal{H})$, there is a finite measure space (X, Ω, μ) and a function $f \in L^\infty(X, \mu)$ such that N is unitary equivalent with M_f, i.e. there is a unitary transformation $U : L^2(X, \mu) \to \mathcal{H}$ such that*

$$N = UM_fU^{-1}.$$

The proof of this theorem uses the concept of cyclic vectors, which we now introduce.

Definition 1.4. Let $A \in \mathcal{B}(\mathcal{H})$. A vector $\xi \in \mathcal{H}$ is called a cyclic vector for A, if $\operatorname{span}\{A^k\xi : k = 0, 1, 2, \dots\}$ is dense in $\mathcal{H}$.

Suppose that N is a normal operator. We prove the spectral theorem in the case when N has a cyclic vector ξ. Define a linear functional $\varphi : C(\sigma(N)) \to \mathbb{C}$ with

$$\varphi(f) = \langle f(N)\xi, \xi\rangle.$$

φ is positive and bounded, since

$$\varphi(|f|^2) = \varphi(\overline{f}f) = \langle f(N)^* f(N)\xi, \xi\rangle = \langle f(N)\xi, f(N)\xi\rangle = \|f(N)\xi\|^2 \geq 0$$

and

$$\Big|\frac{\varphi(f)}{\|f\|}\Big| = \Big|\Big\langle \frac{f(N)}{\|f\|}\xi, \xi\Big\rangle\Big| \leq \Big\|\frac{f(N)}{\|f\|}\xi\Big\|\|\xi\| \leq \|\xi\|^2.$$

Therefore, applying the Riesz representation theorem, there is a unique finite positive Borel measure μ on $\sigma(N)$ such that

$$\int_{\sigma(N)} f d\mu = \langle f(N)\xi, \xi\rangle.$$

Since $C(\sigma(N))$ is dense in $L^2(\sigma(N), \mu)$ and for all $f, g \in C(\sigma(N))$ we have

$$\begin{aligned}\langle f(N)\xi, g(N)\xi\rangle &= \langle g^*(N)f(N)\xi, \xi\rangle = \varphi(\overline{g}f)\\ &= \int_{\sigma(N)} f\overline{g}d\mu = \langle f, g\rangle_{L^2(\sigma(N),\mu)},\end{aligned}$$

the map $C(\sigma(N)) \ni f \to f(N)\xi \in \mathcal{H}$ is an isometry of the dense subspace $C(\sigma(N))$ of $L^2(\sigma(N), \mu)$ onto the dense subspace $\{f(N)\xi : f \in C(\sigma(N))\}$ of $\mathcal{H}$. ($\{f(N)\xi : f \in C(\sigma(N))\}$ is dense in $\mathcal{H}$ because ξ is cyclic.) Therefore it can be uniquely extended to a unitary operator

$$U : L^2(\sigma(N), \mu) \to \mathcal{H}.$$

We shall show that for all $f \in L^2(\sigma(N), \mu)$, we have $UM_f = f(N)U$, where M_f is the multiplication operator given by f. To do this, it is enough to see that for all $g \in C(\sigma(N))$, we have $UM_fg = f(N)Ug$, but this is clear since the definition of U. Therefore, for $\zeta(z) = z$, we have

$$N = UM_\zeta U^{-1},$$

which we needed.

If N does not have a cyclic vector, then we can decompose $\mathcal{H}$ into an orthogonal sum of subspaces H_γ in which there is a cyclic vector for N. To be more precise, we have the following lemma.

Lemma 1.5. *Let $A \in \mathcal{B}(\mathcal{H})$ be a bounded operator. Then there exists a sequence of subspaces $\{H_\gamma\}_{\gamma\in\Gamma}$ such that for each γ, H_γ is invariant for A, it has a cyclic vector and $\mathcal{H} = \sum_{\gamma\in\Gamma} H_\gamma$.*

1.3 Spectral theorem in finite dimensions

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a self-adjoint operator. Suppose that its eigenvalues $\lambda_1, \dots, \lambda_n$ are distinct and the corresponding normalized eigenvectors are denoted with $v_1, \dots, v_n$, and they form an orthonormal basis. Then the *spectral measure at the vector* v is defined as

$$\mu_v = \sum_{k=1}^{n} |\langle v_k, v\rangle|^2 \delta_{\lambda_k}. \tag{1.1}$$

It can be seen that μ_v is characterized by the property that for every polynomial p, we have

$$\langle p(A)v, v\rangle = \int p(x) d\mu_v(x). \tag{1.2}$$

Indeed, since $\{v_1, \dots, v_n\}$ form an orthonormal basis, $v = \sum_{i=1} \langle v_i, v\rangle v_i$. Then

$$\begin{aligned}\langle A^k v, v\rangle &= \sum_{i,j=1}^{n} \lambda_i^k |\langle v_i, v\rangle|^2 |\langle v_j, v\rangle|^2 \langle v_i, v_j\rangle \\ &= \sum_{i=1}^{n} \lambda_i^k |\langle v_i, v\rangle|^2 \\ &= \int x^k d\mu_v(x)\end{aligned}$$

for all $k \in \{0, 1, 2, \dots\}$, but this implies the desired property (1.2). Note that if v is cyclic (for example, $v = \sum_{i=1}^{n} v_k$), μ_v is a spectral measure appearing in Theorem 1.3.

2 Orthogonal polynomials on the real line

In this section, let μ be a finite Borel measure on the real line with compact support K. We assume that K contains infinitely many points in its support and

$$\int x^k d\mu(x) < \infty, \quad k \in \{0, 1, 2, \dots\}.$$

First we state an important lemma.

Lemma 2.1. *For every polynomial* $\Pi(x)$ *for which* $\Pi(x) > 0, x \in K$, *we have*

$$\int \Pi(x) d\mu(x) > 0.$$

The following theorem establishes the existence of orthogonal polynomials.

Theorem 2.2. *There is a unique sequence of polynomials* $\{p_n(x)\}_{n=0}^{\infty}$, $\deg(p_n) \leq n$ *such that*
(i) $p_n(x) = \gamma_n x^n + \dots, \gamma_n > 0$,
(ii) $\int p_n(x) p_m(x) d\mu(x) = \delta_{m,n}$.
The polynomials p_n *are called the orthonormal polynomials with respect to* μ.

Proof. Put $\{1, x, x^2, \dots\}$ to the Gram-Schmidt process. The output will have the required properties. Unicity can be proven indirectly using that every polynomial can be written as a finite linear combination of p_n-s. □

Throughout this text, $p_n(x)$ always denotes the n-th orthonormal polynomial with respect to μ.

Remark 2.3. Since every polynomial Π_n of degree n can be obtained uniquely by a finite linear combination of p_k-s, we have that for $m > n$,

$$\begin{aligned}\int \Pi_n(x)p_m(x)d\mu(x) &= \int p_m(x)\sum_{k=0}^{n} a_k p_k(x)d\mu(x)\\ &= \sum_{k=0}^{n} a_k \int p_k(x)p_m(x)d\mu(x)\\ &= 0.\end{aligned}$$

Now we recall a few known facts from the theory of orthogonal polynomials. First we examine the zeros of p_n and then we shall establish a recurrence formula for them.

Proposition 2.4. *The zeroes of $p_n(x)$ are simple and real.*

Proof. Let $x_1, x_2, \ldots, x_k$ be the real zeroes of $p_n(x)$ with odd multiplicity. Then

$$\int p_n(x)\prod_{l=1}^{k}(x - x_l)d\mu(x) = 0.$$

Since $p_n(x)\prod_{l=1}^{k}(x - x_l)d\mu(x) > 0, x \in K$, it contradicts Lemma 2.1. □

Orthogonal polynomials on the real line are special in some sense. It is possible to obtain a three-term recurrence formula for them, which plays an essential role in their study.

Theorem 2.5. *The polynomials $p_n(x)$ satisfy the recurrence formula*

$$xp_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1}p_{n-1}(x), \quad p_{-1}(x) \equiv 0, \quad b_n > 0. \tag{2.1}$$

Proof. The result can be obtained by observing that

$$\int xp_n(x)p_k(x)d\mu(x) = 0, \quad |k - n| > 1.$$

For positivity of b_n, note that

$$\int xp_{n+1}(x)p_n(x)d\mu(x) = \int \Big(\frac{\gamma_n}{\gamma_{n+1}}p_{n+1}(x) + \Pi_n(x)\Big)p_{n+1}(x)d\mu(x)$$

for some polynomial Π_n of degree at most n, therefore by orthonormality,

$$\int xp_{n+1}(x)p_n(x)d\mu(x) = \frac{\gamma_n}{\gamma_{n+1}} > 0.$$

□

Consider the matrix

$$J_n = \begin{pmatrix} a_0 & b_0 & 0 & \ldots & 0\\ b_0 & a_1 & b_1 & \ldots & 0\\ 0 & b_1 & a_2 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & a_{n-1}\end{pmatrix} \tag{2.2}$$

and the formal infinite matrix

$$J = \begin{pmatrix} a_0 & b_0 & 0 & \ldots\\ b_0 & a_1 & b_1 & \ddots\\ 0 & b_1 & a_2 & \ddots\\ \vdots & \ddots & \ddots & \ddots\end{pmatrix}. \tag{2.3}$$

These are called the Jacobi matrices/operators associated with μ. Define the vector

$$\psi_n(x) := (p_0(x), p_1(x), \ldots, p_{n-1}(x))^T. \tag{2.4}$$

First notice that since J_n is a symmetric matrix, all of its eigenvalues are real. Using (2.1), we see that

$$(J_n - x)\psi_n(x) = (0, \ldots, 0, -b_{n-1}p_n(x))^T.$$

This means that the zeroes of p_n denoted with $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ are the eigenvalues of J_n with the corresponding eigenvectors $\psi_n(x_{i,n})$. Notice that since the eigenvalues are simple, $\{\psi_n(x_{i,n})\}_{i=1}^n$ is an orthogonal basis in $\mathbb{R}^n$. The orthonormal basis given by the eigenvectors is

$$\Big\{\frac{\psi_n(x_{1,n})}{\|\psi_n(x_{1,n})\|}, \ldots, \frac{\psi_n(x_{1,n})}{\|\psi_n(x_{1,n})\|}\Big\}.$$

This way the spectral measure μ_v defined as (1.1) on the real line for the cyclic vector

$$v = \sum_{i=1}^n \frac{\psi_n(x_{i,n})}{\|\psi_n(x_{i,n})\|^2}$$

takes the form

$$\mu_n = \sum_{i=1}^n \frac{1}{\|\psi_n(x_{i,n})\|^2}\delta_{x_{i,n}} = \sum_{i=1}^n \frac{1}{\sum_{k=0}^{n-1} p_k(x_{i,n})^2}\delta_{x_{i,n}}. \tag{2.5}$$

The numbers $(\sum_{k=0}^{n-1} p_k(x_{i,n})^2)^{-1}$ appear frequently in the theory of orthogonal polynomials and they have many forms. For example, with the definition

$$K_n(x, y) := \sum_{k=0}^n p_k(x)p_k(y),$$

we have

$$\frac{1}{K_n(x_0, x_0)} = \inf_{\deg(P_n)\leq n} \int \frac{|P_n(x)|^2}{|P_n(x_0)|^2} d\mu(x).$$

On the other hand, if we choose the cyclic vector

$$w = \sum_{i=1}^n \frac{\psi_n(x_{i,n})}{\sqrt{n}\|\psi_n(x_{i,n})\|},$$

we obtain

$$\nu_n = \sum_{i=1}^n \frac{1}{n}\delta_{x_{i,n}}. \tag{2.6}$$

Under some various conditions imposed on the measure μ, we have $\mu_n \to \mu$ weakly, but using some possibly other conditions, $\nu_n \to \omega$, where ω is the so-called density of states measure.

2.1 Self-adjoint operators and Jacobi matrices

In this subsection we will show that most self-adjoint operators can be regarded as a Jacobi-operator. We say that the self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$ has a *simple spectrum* if for every eigenvalue λ, we have $\dim(\ker(A - \lambda)) = 1$. We study the finite-dimensional case in detail and include a result from the infinite dimensional case.

Proposition 2.6. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible self-adjoint linear transformation with simple spectrum. Then there is a unique orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ such that the matrix of A in this basis is a Jacobi matrix (i.e. it is in the form (2.2)).*

Proof. First we prove that A has a cyclic vector. Since it is self-adjoint with simple spectrum, there is an orthogonal basis formed by its eigenvectors $\{e_1, \dots, e_n\}$. It can be easily seen that $v = e_1 + \dots + e_n$ is a cyclic vector. Indeed, the vectors $v, Av, \dots, A^{n-1}v$ are linearly independent, since

$$A^k v = \sum_{i=1}^{n} \lambda_i^k v_i$$

and the determinant

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i) \neq 0,$$

since all the eigenvalues are distinct and nonzero. Now we put the cyclic basis into the Gram-Schmidt process:

$$\{v_1, v_2, \dots, v_n\} = GS\{v, Av, \dots, A^{n-1}v\}.$$

The vectors e_i satisfy that
(i) $v_i \in \operatorname{span}\{v, Av, \dots, A^{i-1}v\}$,
(ii) $v_i \perp \{v, Av, \dots A^{i-2}v\}$,
(iii) $\langle v_i, A^{i-1}v \rangle > 0$,
(iv) $\|v_i\| = 1$.
From these conditions it can be easily seen that the matrix of A in the basis $\{v_1, \dots, v_n\}$ is in the form (2.2). □

In infinite dimension, we have the following theorem. (See [3], Theorem 7.13.)

Theorem 2.7. *If $A : \mathcal{H} \to \mathcal{H}$ is a self-adjoint transformation which has a cyclic vector, then there is a basis in which its matrix is a Jacobi matrix (i.e. it is in the form (2.3)).*

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