Asymptotics of the Christoffel-Darboux kernel for generalized Jacobi measures

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Orthogonal polynomials

Let μ be a Borel measure on the real line. Assume that $\text{supp}(\mu)$ contains infinitely many points and

$$\int |x|^k d\mu(x) < \infty, \quad k = 0, 1, 2, \ldots$$

Then there exists a unique sequence of polynomials

$$p_n(\mu, x) = p_n(x) = \gamma_n x^n + \ldots, \quad \gamma_n(\mu) = \gamma_n > 0$$

such that

$$\int p_n(x) p_m(x) d\mu(x) = \delta_{n,m}.$$

p_n is called the n-th orthonormal polynomial with respect to μ.

The Christoffel-Darboux kernel

The function

$$K_n(x,y) = \sum_{k=0}^{n} p_k(x)p_k(y)$$

is called the Christoffel-Darboux kernel. K_n has the so-called reproducing property for polynomials, that is, if $\Pi_n(x)$ is an arbitrary polynomial of degree at most n, then

$$\Pi_n(x) = \int \Pi_n(y)K_n(x,y)d\mu(y).$$

The CD kernel plays an important role in, for example,

- approximation theory,
- Fourier analysis,
- random matrix theory, especially universality limits.

Universality limits and random matrices

If the eigenvalue distribution of a $n \times n$ Hermitian unitary ensemble is given by

$$p(x_1, \dots, x_n) = \frac{1}{Z^n} \prod_{1 \leq i < j \leq n} |x_i - x_j|^2 \prod_{k=1}^{N} w(x_k) dx_k,$$

the k-point correlation function defined by

$$R_{k,n}(x_1, \dots, x_k) = \frac{n!}{(n-k)!} \int \dots \int p(x_1, \dots, x_n) dx_{k+1} \dots dx_n$$

can be expressed as

$$R_{k,n}(x_1, \dots, x_k) = \det \left(w(x_i)^{1/2} w(x_j)^{1/2} K_n(x_i, x_j) \right)_{i,j=1}^{n},$$

where $K_n(x, y)$ is the Christoffel-Darboux kernel with respect to the measure $d\mu(x) = w(x)dx$. ($w(x)$ is given in the eigenvalue distribution.)

Universality limits and random matrices

In particular, $R_{k,n}(x_1, \dots, x_k)$ is interesting for fixed k and large n. Since

$$R_{k,n}(x_1, \dots, x_k) = \det \left(w(x_i)^{1/2} w(x_j)^{1/2} K_n(x_i, x_j) \right)_{i,j=1}^{n},$$

the local behavior of the eigenvalue distribution is described by the limit

$$\lim_{n \to \infty} \frac{K_n(x_0 + \frac{a}{n}, x_0 + \frac{b}{n})}{K_n(x_0, x_0)},$$

which are called universality limits.

Known results

Theorem (D. S. Lubinsky)

Let μ be a finite Borel measure such that

- μ is supported on $[-1, 1]$,
- μ is regular in the sense of Stahl-Totik (i.e. $\gamma_n(\mu)^{1/n} \to 2$, where $\gamma_n(\mu)$ is the leading coefficient of $p_n(\mu, x)$),
- for some $x_0 \in (-1, 1)$ the measure μ is absolutely continuous in a small neighborhood of x_0 with $d\mu(x) = w(x)dx$ there for some *continuous and strictly positive* weight w.

Then we have

$$\lim_{n\to\infty} \frac{K_n(x_0 + \frac{a}{n}, x_0 + \frac{b}{n})}{K_n(x_0, x_0)} = \frac{\sin \pi\omega_{[-1,1]}(x_0)(b - a)}{\pi\omega_{[-1,1]}(x_0)(b - a)},$$

where $\omega_{[-1,1]}(x) = (\pi\sqrt{1 - x^2})^{-1}$ is the density of the equilibrium measure for $[-1, 1]$.

Generalized Jacobi measures

What happens when the measure is not that nice locally?

We call (during this talk) μ a *generalized Jacobi measure*, if it is

- supported on a compact set K,
- regular in the sense of Stahl-Totik ($\gamma(\mu)^{1/n} \to 1/\operatorname{cap}(\operatorname{supp}(\mu))$),
- for some $x_0 \in K$ the measure μ is absolutely continuous in an interval around x_0 (or an interval with x_0 as an endpoint if x_0 is an endpoint of K), and we have

  $$d\mu(x) = w(x)|x - x_0|^\alpha dx$$

  there for some $\alpha > -1$, where w is *positive and continuous* (or left/right continuous, when x_0 is an endpoint).

More known results

Theorem (D. S. Lubinsky)

Let μ be a generalized Jacobi measure with compact support K, and suppose that the singularity at x_0 is a *right endpoint* of K. Then if

$$\lim_{n \to \infty} \frac{K_n(x_0 - a\eta_n, x_0 - a\eta_n)}{K_n(x_0, x_0)} = \frac{\mathbb{J}^*_\alpha(a, a)}{\mathbb{J}^*_\alpha(0, 0)},$$

holds for each nonnegative a (where $\eta_n = O(n^{-2})$ is a given sequence), we have

$$\lim_{n \to \infty} \frac{K_n(x_0 - a\eta_n, x_0 - b\eta_n)}{K_n(x_0, x_0)} = \frac{\mathbb{J}^*_\alpha(a, b)}{\mathbb{J}^*_\alpha(0, 0)},$$

uniformly for a, b in compact subsets of the complex plane... *(continued on the next slide)*

More known results

Theorem (continued from the previous slide)

...where $\mathbb{J}^*_\alpha(a, b)$ is the "entire-ized" version of the Bessel kernel defined by

$$\mathbb{J}^*_\alpha(a, b) = \frac{1}{a^{\alpha/2} b^{\alpha/2}} \frac{J_\alpha(\sqrt{a})\sqrt{b}J'_\alpha(\sqrt{b}) - J_\alpha(\sqrt{b})\sqrt{a}J'_\alpha(\sqrt{a})}{2(a-b)}$$

and $J_\nu(z)$ is the Bessel function of the first kind and order ν.

Questions.

- Does the condition hold in the second theorem of Lubinsky?
- What happens when the singularity x_0 is in the interior of K?

New results

Does the condition hold in the second theorem of Lubinsky? Yes, it does!

Theorem, endpoint case (D.)

Let μ be a generalized Jacobi measure with compact support K, and suppose that the singularity at x_0 is a *right endpoint* of K, moreover we have

$$d\mu(x) = w(x)|x - x_0|^\alpha dx, \quad (x_0 - \varepsilon, x_0]$$

there for some positive and continuous w and α ¿ -1. Then

$$\lim_{n\to\infty} \frac{1}{n^{2\alpha+2}} K_n\Big(x_0 - \frac{a}{2n^2}, x_0 - \frac{a}{2n^2}\Big) = 2^{\alpha+1}\frac{M(K,x_0)^{2\alpha+2}}{w(x_0)}\mathbb{J}^*_\alpha\big(M(K,x_0)^2 a, M(K,x_0)^2 a\big)$$

holds for all nonnegative a, where $M(K,x_0) = \lim_{x\to x_0-} \sqrt{2}\pi|x - x_0|^{1/2}\omega_K(x)$.

Corollary: using Lubinsky's second theorem, it follows immediately that the same holds after replacing the a in the second arguments with b.

More new results

What happens when the singularity x_0 is in the interior of K?

Theorem, interior point case (D.)

Let μ be a generalized Jacobi measure with compact support K, and suppose that the singularity at x_0 is *in the interior* of K, moreover we have

$$d\mu(x) = w(x)|x - x_0|^\alpha dx, \quad (x_0 - \varepsilon, x_0 + \varepsilon)$$

there for some positive and continuous w and α ¿ -1. Then

$$\lim_{n\to\infty} \frac{1}{n^{\alpha+1}} K_n\Big(x_0 + \frac{a}{n}, x_0 + \frac{a}{n}\Big) = \frac{(\pi\omega_K(x_0))^{\alpha+1}}{w(x_0)} \mathbb{L}_\alpha^*(\pi\omega_K(x_0)a, \pi\omega_K(x_0)a)$$

holds for all real a and as consequence,

$$\lim_{n\to\infty} \frac{K_n(x_0 + \frac{a}{n}, x_0 + \frac{b}{n})}{K_n(x_0, x_0)} = \frac{\mathbb{L}_\alpha^*(\pi\omega_K(x_0)a, \pi\omega_K(x_0)b)}{\mathbb{L}_\alpha^*(0,0)},$$

holds uniformly for a, b in compact subsets of the complex plane, where...
(continued on the next slide)

More new results

Theorem (continued from the previous slide)

...the kernel function $\mathbb{L}_\alpha^*(a,b)$ is defined by

$$\mathbb{L}_\alpha^*(a,b) = \frac{1}{a^{\frac{\alpha-1}{2}} b^{\frac{\alpha-1}{2}}} \frac{J_{\frac{\alpha+1}{2}}(a) J_{\frac{\alpha-1}{2}}(b) - J_{\frac{\alpha+1}{2}}(b) J_{\frac{\alpha-1}{2}}(a)}{2(a-b)}$$

and $\omega_K(x)$ is the density of the equilibrium measure for K.

The long proof in three sentences:

- Prove the theorems for the special measures $|x-1|^\alpha dx$ and $|x|^\alpha dx$ supported on $[-1,1]$ using the Riemann-Hilbert method.
- Use the polynomial inverse image method of Totik to extend the asymptotics along the diagonal for generalized Jacobi measures.
- Apply the entire functions approach of Lubinsky to prove universality limits from asymptotics along the diagonal.

Some open questions

So far, we have imposed two important conditions upon the measure:

- A global condition: Stahl-Totik regularity $(\gamma(\mu)^{1/n} \to 1/\operatorname{cap}(\operatorname{supp}(\mu)))$
- A local condition: behaving like $|x - x_0|^\alpha dx$ around x_0

What happens when we omit or modify some of them?

Without Stahl-Totik regularity: we cannot even hope for the existence of

$$\lim_{n\to\infty} \frac{1}{n} K_n(x_0, x_0)$$

like in the cases studied now, *but*

$$\lim_{n\to\infty} \frac{K_n(x_0 + \frac{a}{n}, x_0 + \frac{b}{n})}{K_n(x_0, x_0)}$$

may exist!

Some open questions

Without a local condition: without any, we cannot expect to describe the universality limits (we know that it will be a reproducing kernel for a De Branges space but there are a lot of such kernels), but by modifying the local condition, for example if we have a Fisher-Hartwig type singularity

$$d\mu(x) = |x - x_0|^\alpha \Theta(x) dx,$$

where $\Theta(x)$ is a pure jump function

$$\Theta(x) = \begin{cases} A & \text{if } x \leq x_0, \\ B & \text{if } x > x_0, \end{cases}$$

describing the limit is still open and interesting.

References I

A. B. J. Kuijlaars and M. Vanlessen
Universality for eigenvalue correlations from the modified Jacobi unitary ensemble
Int. Math. Res. Not., Vol. 30 (2002)

D. S. Lubinsky
A new approach to universality limits involving orthogonal polynomials
Annals of Mathematics, Vol. 170 (2009)

D. S. Lubinsky
Universality limits at the hard edge of the spectrum for measures with compact support
Int. Math. Res. Not., Art. ID rnn 099 (2008)

B. Simon
Two extensions of Lubinsky's universality theorem
J. d'Analyse Math. 105 (2008)

References II

V. Totik
Universality and fine zero spacing on general sets
Arkiv för Matematik, Vol. 47 (2009)

V. Totik
Christoffel functions on curves and domains
Transactions of the AMS, Vol. 362 (2010)

V. Totik
Asymptotics of Christoffel functions on arcs and curves
Advances in Mathematics, Vol. 252 (2014)

T. Danka and V. Totik
Christoffel functions with power type weights
to appear in Journal of the European Mathematical Society

T. Danka
Universality limits for generalized Jacobi measures
submitted

Thank you for your attention!