Asymptotics of the Christoffel-Darboux kernel for generalized Jacobi measures

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Orthogonal polynomials

Let μ be a Borel measure on the complex plane. Assume that $\mathrm{supp}(\mu)$ contains infinitely many points and

$$\int |z|^k d\mu(z) < \infty, \quad k = 0, 1, 2, \dots$$

Then there exists a unique sequence of polynomials

$$p_n(z,\mu) = p_n(z) = \gamma_n z^n + \dots, \quad \gamma_n(\mu) = \gamma_n > 0$$

such that

$$\int p_n(z)\overline{p_m(z)} d\mu(z) = \delta_{n,m}.$$

p_n is called the n-th orthonormal polynomial with respect to μ.

The Christoffel-Darboux kernel

The function

$$K_n(z, w) = \sum_{k=0}^{n} p_k(z)\overline{p_k(w)}$$

is called the Christoffel-Darboux kernel. K_n has the so-called reproducing property for polynomials, that is, if $\Pi_n(z)$ is an arbitrary polynomial of degree at most n, then

$$\Pi_n(z) = \int \Pi_n(w) K_n(z, w) d\mu(w).$$

The CD kernel plays an important role in, for example,

- approximation theory,
- Fourier analysis,
- random matrix theory, especially universality limits.

Universality limits

Suppose now that μ is supported on the real line. Limits of the type

$$\lim_{n\to\infty} \frac{K_n(x_0 + a/n, x_0 + b/n)}{K_n(x_0, x_0)}$$

are called *universality limits*, they are especially important in random matrix theory.

Up until the landmark paper 'A New Approach to Universality Limits Involving Orthogonal Polynomials' by D. S. Lubinsky (published in the Annals of Mathematics), existence of universality limits were known under very strong conditions imposed upon the measure, for example analiticity of the weight $d\mu(x)/dx$ on the whole support.

Lubinsky's result

D. S. Lubinsky proved the following.

Theorem (Lubinsky, 2009)

Let μ be a finite Borel measure with

- $\text{supp}(\mu) = [-1, 1]$
- μ is regular in the sense of Stahl and Totik, that is $\lim_{n\to\infty} \gamma_n(\mu)^{1/n} = \frac{1}{\text{cap}([-1,1])} = 2$ holds.

Also suppose that for some $x_0 \in [-1, 1]$, μ is absolutely continuous in a small neighbourhood $(x_0 - \varepsilon, x_0 + \varepsilon)$ and

$$d\mu(x) = w(x)dx, \quad x \in (x_0 - \varepsilon, x_0 + \varepsilon)$$

there, where $w(x)$ is **continuous and positive** at x_0. Then

$$\lim_{n\to\infty} \frac{K_n(x_0 + a/n, x_0 + b/n)}{K_n(x_0, x_0)} = \frac{\sin \pi\omega_{[-1,1]}(x_0)(b-a)}{\pi\omega_{[-1,1]}(x_0)(b-a)},$$

where $\omega_{[-1,1]}(x) = (\pi\sqrt{1-x^2})^{-1}$ is the density of the equilibrium measure.

The essence of Lubinsky's result

Lubinsky's theorem was generalized for Borel measures with compact support by B. Simon and V. Totik simultaneously. They used very different methods.

Lubinsky's method has three main steps.

Step 1. Study universality limits along the diagonal $a = b$. Along the diagonal we have

$$\frac{1}{K_n(x_0, x_0)} = \lambda_n(\mu, x_0) = \inf_{\deg(P_n) \leq n} \int \frac{|P_n(x)|^2}{|P_n(x_0)|^2} d\mu(x),$$

where the infimum is taken for polynomials of degree at most n. These are called Christoffel functions and

- their asymptotics are fairly well known,
- moreover they have some very powerful localization properties, which can be exploited.

The essence of Lubinsky's result

Step 2. For two measures $\mu \leq \mu^*$ compare the asymptotic behavior of the CD kernel with the inequality

$$\frac{|K_n(x,y,\mu) - K_n(x,y,\mu^*)|}{K_n(x,x,\mu)} \leq \left(\frac{\lambda_n(\mu,x)}{\lambda_n(\mu,y)}\right)^{1/2} \left(1 - \frac{\lambda_n(\mu,x)}{\lambda_n(\mu^*,x)}\right)^{1/2},$$

which is called Lubinsky's inequality.

Step 3. Using Christoffel functions show that if a measure behaves like the Lebesgue measure in a small neighborhood of x_0, then their universality limits are the same. For the Lebesgue measure, we have universality.

Generalized Jacobi measures

Let μ be a finite Borel measure and suppose that its support is a compact subset K of the real line.
Let $x_0 \in K$ and suppose that

- x_0 is a right endpoint of K, that is $K \cap (x_0, x_0 + \varepsilon) = \varnothing$ for some small $\varepsilon > 0$,
- μ is regular in the sense of Stahl and Totik,
- μ is absolutely continuous in a small interval $[x_0 - \delta, x_0]$ with

$$d\mu(x) = w(x)|x - x_0|^\alpha dx, \quad x \in [x_0 - \delta, x_0]$$

  there for some $\alpha > -1$ and continuous, positive $w(x)$.

We will call such measures *generalized Jacobi measures*.

Also, define the so-called (”entire-ized”) Bessel kernel as

$$\mathbb{J}_\alpha^*(a, b) = \frac{1}{a^{\alpha/2} b^{\alpha/2}} \frac{J_\alpha(\sqrt{a})\sqrt{b} J_\alpha'(\sqrt{b}) - J_\alpha(\sqrt{b})\sqrt{a} J_\alpha'(\sqrt{a})}{2(a-b)},$$

where J_ν denotes the Bessel function of the first kind and order ν.

Lubinsky's result for generalized Jacobi measures

Theorem (Lubinsky, 2009)

For the generalized Jacobi measure μ defined as before, if

$$\lim_{n\to\infty} \frac{K_n(1-\frac{a^2}{2n^2}, 1-\frac{a^2}{2n^2})}{K_n(1,1)} = \frac{\mathbb{J}_\alpha^*(a^2,a^2)}{\mathbb{J}_\alpha^*(0,0)}$$

holds for all $a \in \mathbb{R}$, then

$$\lim_{n\to\infty} \frac{K_n(1-\frac{a^2}{2n^2}, 1-\frac{b^2}{2n^2})}{K_n(1,1)} = \frac{\mathbb{J}_\alpha^*(a^2,b^2)}{\mathbb{J}_\alpha^*(0,0)}$$

holds uniformly for a, b in compact subsets of $\mathbb{C}$.

Question. Does the former condition hold?

New results

Yes, it does!

Theorem (D.)

For the generalized Jacobi measure μ defined before,

$$\lim_{n\to\infty} \frac{1}{n^{2\alpha+2}} K_n\left(x_0 - \frac{a}{2n^2}, x_0 - \frac{a}{2n^2}\right) = \frac{M(K, x_0)^{2\alpha+2}}{w(x_0)} 2^{\alpha+1} \mathbb{J}^*_\alpha\left(M(K, x_0)^2 a, M(K, x_0)^2 a\right)$$

holds for all $a \in [0, \infty)$, where $M(K, x_0) = \lim_{x\to x_0} \sqrt{2}\pi |x - x_0|^{1/2} \omega_K(x_0)$ and $\omega_K(x)$ is the density of the equilibrium measure for K.

Corollary

Using Lubinsky's result, we have

$$\lim_{n\to\infty} \frac{K_n(1 - \frac{a^2}{2n^2}, 1 - \frac{b^2}{2n^2})}{K_n(1, 1)} = \frac{\mathbb{J}^*_\alpha(a^2, b^2)}{\mathbb{J}^*_\alpha(0, 0)}, \quad a, b \in \mathbb{C}.$$

Sketch of the proof

The proof uses the polynomial inverse image method of Totik.

Step 1. Consider the Jacobi measure

$$d\mu_\alpha(x) = |1 - x|^\alpha dx, \quad x \in [-1, 1].$$

Asymptotics for K_n around 1 is well known.

Step 2. Take a polynomial $T_N(x)$ and calculate the asymptotics of K_n along the diagonal for the pullback measure on $T_N^{-1}([-1, 1])$. The set $T_N^{-1}([-1, 1])$ is the union of intervals.

Step 3. (The hard part.) Approximate the support of μ with sets of the form $T_N^{-1}([-1, 1])$ and approximate μ *locally* with pullback measures of μ_α.

Step 4. Localize and calculate the asymptotics for K_n for μ using results for pullback measures.

Questions and partial answers

What happens when the point x_0 is not an endpoint of the support of μ?

In this case, the asymptotics along the diagonal are also known.

Theorem (D.)

If x_0 is in the interior of the support of the generalized Jacobi measure, then

$$\lim_{n\to\infty} \frac{1}{n^{\alpha+1}} K_n(x_0 + a/n, x_0 + a/n) = \frac{(\pi\omega_K(x_0))^{\alpha+1}}{w(x_0)} \mathbb{L}^*_\alpha(\pi\omega_K(x_0)a, \pi\omega_K(x_0)a),$$

where

$$\mathbb{L}^*_\alpha(a,b) = \frac{1}{a^{\frac{\alpha-1}{2}} b^{\frac{\alpha-1}{2}}} \frac{J_{\frac{\alpha+1}{2}}(a) J_{\frac{\alpha-1}{2}}(b) - J_{\frac{\alpha+1}{2}}(b) J_{\frac{\alpha-1}{2}}(a)}{2(a-b)},$$

which is a kernel function I was unable to find anywhere in the literature.

Questions and partial answers

Would the approach of Lubinsky's second theorem (convergence along the diagonal $\iff$ convergence along offdiagonal) work for the case when the singularity is an interior point?

I hope so.

So far, I only managed to show the following theorem using the Riemann-Hilbert method.

Proposition

Let μ_α be the measure defined as

$$d\mu_\alpha(x) = |x|^\alpha dx, \quad x \in [-1, 1].$$

Then

$$\frac{1}{n} K_n(a/n, b/n) = \mathbb{L}^*_\alpha(a, b) + O\left(\frac{1}{n}\right),$$

where $O(1/n)$ is uniform for compact subsets of the real line.

Questions and partial answers

Conjecture

For generalized Jacobi measures with the singularity inside the support, we have

$$\lim_{n\to\infty} \frac{K_n(x_0 + a/n, x_0 + b/n)}{K_n(x_0, x_0)} = \frac{\mathbb{L}^*_\alpha(\pi\omega_K(x_0)a, \pi\omega_K(x_0)b)}{\mathbb{L}^*_\alpha(0,0)}$$

uniformly for a, b in compact subsets of $\mathbb{C}$.

Questions and partial answers

If the conjecture would be true, it would have some nice applications, for example it would probably imply fine zero spacing of orthogonal polynomials. With this, I hope to sharpen the following theorem.

Theorem (Last, Simon, 2008)

Suppose that μ is supported on a compact subset of the real line, and for some x_0, we have $d\mu(x) = w(x)dx$ in a small neighborhood of x_0 with

$$0 < \liminf_{x \to x_0} \frac{w(x)}{|x - x_0|^\alpha} \leq \limsup_{x \to x_0} \frac{w(x)}{|x - x_0|^\alpha} < \infty$$

for some $\alpha > -1$. Then

$$\limsup_{n \to \infty} n |x_{-1,n}(x_0) - x_{1,n}(x_0)| < \infty,$$

where

$$\dots < x_{-1,n}(x_0) < x_0 \leq x_{1,n}(x_0) < \dots$$

are the zeros of the n-th orthonormal polynomial $p_n(x, \mu)$ ordered around x_0.

What happens when the support of the generalized Jacobi measure is not a subset of the real line?

Let μ be a finite Borel measure on the complex plane and assume that

- μ is regular in the sense of Stahl and Totik,
- the support $\gamma := \text{supp}(\mu)$ consists of finitely many Jordan curves and arcs lying exterior to each other.

Generalized Jacobi measures on Jordan curves and arcs I

Suppose that z_0 is in the two dimensional interior of γ and

- γ is C^2-smooth in a neighbourhood of z_0,
- $d\mu(z) = w(z)|z - z_0|^\alpha ds_\gamma(z), \alpha > -1$ for some continuous and positive weight $w(z)$ on that neighbourhood, where s_γ denotes the arc-length measure of γ.

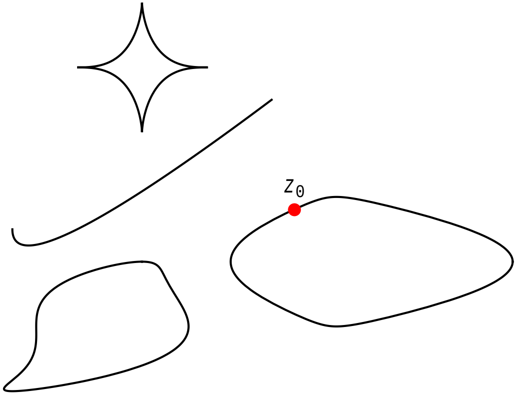
Theorem (Totik, D., 2015)

For the μ defined previously we have

$$\lim_{n\to\infty} \frac{1}{n^{\alpha+1}} K_n(z_0, z_0) = \frac{(\pi\omega_\gamma(z_0))^{\alpha+1}}{2^{\alpha+1} w(z_0)} \left(\Gamma\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha+3}{2}\right)\right)^{-1},$$

where ω_γ denotes the density of the equilibrium measure for γ.

Generalized Jacobi measures on Jordan curves and arcs I



A typical situation where the previous theorem can be applied

Generalized Jacobi measures on Jordan curves and arcs II

Now suppose that the support of μ consists of finitely many Jordan curves and arcs lying exterior to each other **and z_0 is an endpoint for one of the arcs**.

Theorem (Totik, D., 2015)

With the previous notations and conditions, we have

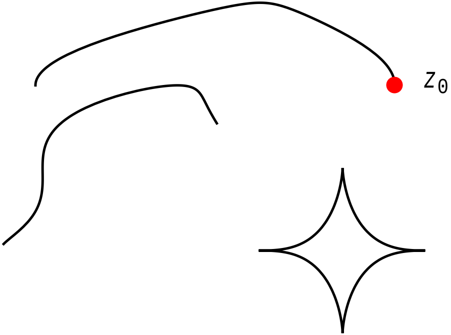
$$\lim_{n\to\infty} \frac{1}{n^{2\alpha+2}} K_n(z_0, z_0) = \frac{(\pi M(\gamma, z_0))^{2\alpha+2}}{2^{\alpha+1} w(z_0)} \Big(\Gamma(\alpha+1)\Gamma(\alpha+2)\Big)^{-1},$$

where

$$M(\gamma, z_0) := \lim_{z\to z_0, z\in\gamma} \sqrt{2}\pi |z - z_0|^{1/2} \omega_\gamma(z_0)$$

is the behaviour of the equilibrium density near the endpoint z_0.

Generalized Jacobi measures on Jordan curves and arcs II



A typical situation where the previous theorem can be applied

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Thank you for your attention!