Christoffel functions on Jordan arcs and curves with power type weights

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Christoffel functions

Let μ be a finite Borel measure on $\mathbb{C}$ such that

- $K := \text{supp}(\mu)$ is compact and
- K contains infinitely many points.

Definition (Christoffel functions)

The n-th *Christoffel function* associated with μ is defined as

$$\lambda_n(\mu, z_0) := \inf_{\deg(P_n) \leq n} \int \frac{|P_n(z)|^2}{|P_n(z_0)|^2} d\mu(z),$$

where the infimum is taken for all polynomials of degree at most n.

Christoffel functions

$p_n(z, \mu) = p_n(z)$: n-th orthonormal polynomial w.r.t. μ
$K_n(z, w, \mu) = K_n(z, w)$: Christoffel-Darboux kernel defined as

$$K_n(z, w) = \sum_{k=0}^{n} \overline{p_k(z)} p_k(w).$$

Then it is known that

$$\lambda_n(\mu, z_0) = \frac{1}{K_n(z_0, z_0)} = \frac{1}{\sum_{k=0}^{n} |p_k(z_0)|^2}.$$

Applications of Christoffel functions include

- approximation theory,
- Fourier analysis,
- reproducing kernels,
- random matrix theory, especially universality limits.

Szegő obtained the following theorem in the early 20th century.

Theorem (Szegő)

Let μ be a finite Borel measure supported on the unit circle $\mathbb{T}$ such that

- μ is a.c. with $d\mu(e^{it}) = w(e^{it})dt$ and
- $\frac{1}{2\pi}\int_{-\pi}^{\pi} \log w(e^{it})dt > -\infty$ holds. (This is called the Szegő condition.)

Then

$$\lim_{n\to\infty} \lambda_n(\mu, z) = (1-|z|^2)\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{Re}\left[\frac{e^{it}+z}{e^{it}-z}\right]\log w(e^{it})dt\right), \quad |z| < 1.$$

This theorem influenced the theory of Hardy spaces and motivated to study the asymptotics for orthogonal polynomials.

Szegő's theorem

What about points on the unit circle?

Theorem

Let μ be a finite Borel measure supported on the unit circle with

$$d\mu(e^{it}) = w(e^{it})dt + d\mu_s(e^{it}), \quad t \in [-\pi, \pi).$$

Then

$$\lim_{n\to\infty} \lambda_n(\mu, z_0) = \mu(\{z_0\}), \quad |z_0| = 1.$$

In this case, we want to obtain information about the rate of convergence.

Theorem of A. Máté, P. Nevai and V. Totik

Theorem (Máté, Nevai, Totik, 1991)

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If the Szegő condition

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log w(e^{it})dt > -\infty$$

holds, then

$$\lim_{n\to\infty} n\lambda_n(\mu, e^{it}) = 2\pi w(e^{it}) \quad t \in [-\pi, \pi) \text{ almost everywhere.}$$

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Theorems of these type can be generalized in two ways.

- Study measures with more general support.
- Study measures supported on the unit circle, but with weaker conditions.

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- Study measures with more general support.
- Study measures supported on the unit circle, but with weaker conditions.

(Or study measures with more general support and weaker conditions.)

A local version of the Máté-Nevai-Totik theorem

The previous theorem of Máté, Nevai and Totik has a local version, which requires a *local* and a *global* condition on μ.

Definition (Regularity in the Stahl-Totik sense)

μ is said to be regular in the sense of Stahl and Totik, or $\mu \in \mathbf{Reg}$, if for every sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ we have

$$\limsup_{n \to \infty} \left(\frac{|P_n(z)|}{\|P_n\|_{L^2(\mu)}} \right)^{1/\deg(P_n)} \leq 1, \quad z \in \operatorname{supp}(\mu).$$

A local version of the Máté-Nevai-Totik theorem

The local version of the previous theorem of Máté, Nevai and Totik says the following.

Theorem (Máté, Nevai, Totik, 1991)

Let μ be a finite Borel measure **regular in the sense of Stahl-Totik** supported on the unit circle with

$$d\mu(e^{it}) = w(e^{it})dt + d\mu_s(e^{it}), \quad t \in [-\pi, \pi).$$

If for some interval $I \subseteq [-\pi, \pi)$ the **local** Szegő condition

$$\frac{1}{2\pi} \int_I \log w(e^{it}) dt > -\infty$$

holds, then

$$\lim_{n \to \infty} n\lambda_n(\mu, e^{it}) = 2\pi w(e^{it}) \quad t \in I \text{ almost everywhere.}$$

Extensions of the Máté-Nevai-Totik theorem

Generalization is possible in two directions:

- more general support,
- more general conditions on μ.

A partial list of mighty results:

- 1991, A. Máté, P. Nevai and V. Totik: support on $\mathbb{T}$; global Szegő condition or (local Szegő condition + Stahl-Totik regularity)
- 2000, V. Totik: support on a general compact set of $\mathbb{R}$; local Szegő condition + Stahl-Totik regularity
- 2008, D. S. Lubinsky: comparison method for measures on $[-1,1]$
- 2009, B. Simon and V. Totik (separately): extension of Lubinsky's result for measures supported on a general compact set of $\mathbb{R}$
- 2010, V. Totik: support on a set of disjoint Jordan curves; continuity AND positivity of the weight w at a fixed point + Stahl-Totik regularity
- 2014, V. Totik: support on a set of disjoint Jordan arcs and curves; continuity AND positivity of the weight w at a fixed point + Stahl-Totik regularity

Concepts from potential theory

To talk about measures with more general support, we need some concepts from potential theory.

Let μ be an arbitrary Borel measure on the complex plane. The *energy* of μ is defined as

$$I(\mu) := \int \int \log \frac{1}{|z - w|} d\mu(z) d\mu(w).$$

The energy of the *set* K is defined as

$$I(K) := \inf_{\mu \in \mathcal{M}_1(K)} I(\mu),$$

where $\mathcal{M}_1(K)$ denotes the probability measures supported on K.

Equilibrium measure

Let $K \subseteq \mathbb{C}$ be a set with $I(K) > -\infty$.

Definition (equilibrium measure)

If ν_K is a probability measure supported on K such that

$$I(\nu_K) = I(K),$$

then ν_K is called an *equilibrium measure* of K.

If K is compact then there is an unique equilibrium measure.

Example 1. The equilibrium measure for the unit circle $\mathbb{T}$ is the normed Lebesgue measure

$$d\nu_{\mathbb{T}}(e^{it}) = \frac{1}{2\pi} dt, \quad t \in [-\pi, \pi).$$

Example 2. The equilibrium measure for the interval $[-1, 1]$ is the Chebyshev-distribution

$$d\nu_{[-1,1]}(x) = \frac{1}{\pi\sqrt{1 - x^2}} dx, \quad x \in [-1, 1].$$

The main result, I

Let μ be a finite Borel measure on the complex plane and assume that

- $\mu \in \mathbf{Reg}$,
- the support $\gamma := \text{supp}(\mu)$ consists of finitely many Jordan curves and arcs lying exterior to each other.

Suppose that z_0 is in the two dimensional interior of γ and

- γ is C^2-smooth in a neighbourhood of z_0,
- $d\mu(z) = w(z)|z - z_0|^\alpha ds_\gamma(z), \alpha > -1$ for some continuous and positive weight $w(z)$ on that neighbourhood, where s_γ denotes the arc-length measure of γ.

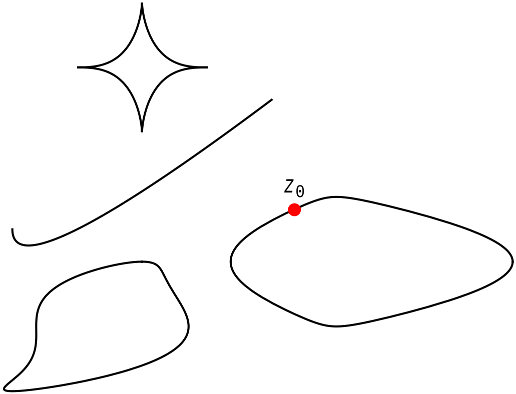
Theorem (Totik, D., 2015)

With the previous notations and conditions, we have

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\mu, z_0) = \frac{w(z_0)}{(\pi\omega_\gamma(z_0))^{\alpha+1}} 2^{\alpha+1}\Gamma\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha+3}{2}\right),$$

where ω_γ denotes the density of the equilibrium measure for γ.

The main result, I



A typical situation where the previous theorem can be applied

The main result, II

Now suppose that the support of μ consists of finitely many Jordan curves and arcs lying exterior to each other **and z_0 is an endpoint for one of the arcs**.

Theorem (Totik, D., 2015)

With the previous notations and conditions, we have

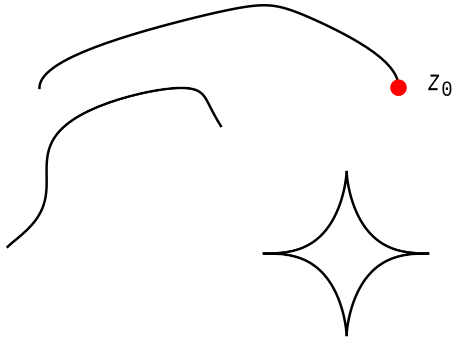
$$\lim_{n\to\infty} n^{2\alpha+2}\lambda_n(\mu, z_0) = \frac{w(z_0)}{(\pi M(\gamma, z_0))^{2\alpha+2}}\Gamma(\alpha+1)\Gamma(\alpha+2),$$

where

$$M(\gamma, z_0) := \lim_{z\to z_0, z\in\gamma} \sqrt{|z-z_0|}\omega_\gamma(z_0)$$

is the behaviour of the equilibrium density near the endpoint z_0.

The main result, II



A typical situation where the previous theorem can be applied

Future work

It would be nice to know: does something stronger, like

$$\lim_{n\to\infty} \frac{\lambda_n(\mu, z_0 + a/n)}{\lambda_n(\mu, z_0)} = 1$$

holds uniformly for a in compact subsets of $\mathbb{C}$? (These type of limits are strongly connected to the so-called universality limits in random matrix theory.)
Work in progress!

It would be very nice to know: can the Szegő condition $\int_{-\pi}^{\pi} \log w(e^{it}) dt > -\infty$ be replaced in the Máté-Nevai-Totik theorem for $w(e^{it}) > 0$ Lebesgue-almost everywhere? That is, does

$$\lim_{n\to\infty} n\lambda_n(\mu, e^{it}) = 2\pi w(e^{it}), \quad t \in [-\pi, \pi) \text{ almost everywhere}$$

holds with the much more weaker $w(e^{it}) > 0$ a.e. condition?

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T. Danka and V. Totik
Christoffel functions with power type weights
submitted for consideration for publication

Thank you for your attention!