Christoffel functions on Jordan curves with respect to measures with jump singularity

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Definition (Christoffel functions)

The n-th Christoffel function with respect to μ is defined by

$$\lambda_n(\mu, z_0) = \inf_{\deg(P_n) \leq n} \int \frac{|P_n(z)|^2}{|P_n(z_0)|^2} d\mu(z),$$

where the infimum is taken for all polynomials P_n such that $\deg(P_n) \leq n$.

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Goal: to study the asymptotic behavior of $\lambda_n(\mu, z_0)$, if μ is supported on a Jordan curve and has a jump singularity at z_0.

Previous results: $\mathrm{supp}(\mu) = [-1, 1]$

Theorem (A. Foulquié Moreno, A. Martínez-Finkelshtein, V. L. Sousa, 2011)

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The aim of this talk is to extend this theorem for weights supported on a Jordan curve system lying in the complex plane.

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Definition (Equilibrium measure)

$\nu_K \in \mathcal{P}(K)$ is called an equilibrium measure, if

$$I(\nu_K) = \sup_{\mu \in \mathcal{P}(K)} I(\mu)$$

holds.

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Then we have

$$\lim_{n \to \infty} n\lambda_n(\mu, x_0) = \frac{1}{\omega_S(x_0)} \frac{A - B}{\log A - \log B},$$

where ω_S is the weight function of the equilibrium measure of S.

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Theorem (V. Totik, 2012)

With the previously introduced notations and assumptions, we have

$$\lim_{n\to\infty} n\lambda_n(\mu, z_0) = \frac{w(z_0)}{\omega_\gamma(z_0)}.$$

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Remark 1. $\omega_\gamma(z_0)$ and therefore the asymptotic limit can be written in terms of Green's functions.

Remark 2. Taking $B = A + \varepsilon$ and letting $\varepsilon \to 0$ yields the continuous case.

Remark 3. The asymptotic behavior depends only on the set and the local behavior of w.

The main result

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The *Erdős-Turán* condition tells us that $w > 0$ implies

$$\limsup_{n\to\infty} \left(\frac{\|P_n\|_\gamma}{\|P_n\|_2} \right)^{1/n} \leq 1,$$

if P_n is a polynomial of degree n. This means that $\|P_n\|_2$ is *not* exponentially larger than $\|P_n\|_\gamma$.

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if P_n is a polynomial of degree n. This means that $\|P_n\|_2$ is *not* exponentially larger than $\|P_n\|_\gamma$.

In the proof, we are going to use the method of *polynomial inverse images*.

Outline of the proof

1. **Transfer the known results for the interval $[-1, 1]$ to the unit circle.**

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$$d\mu_{\mathbb{T}}(e^{it}) = v(e^{it})dt,$$

where

$$v(e^{it}) = \begin{cases} A & \text{if } -\pi/2 \leq t \leq \pi/2, \\ B & \textit{otherwise.} \end{cases}$$

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By transfering $\mu_{\mathbb{T}}$ to a measure on $[-1,1]$ via the mapping $e^{it} \mapsto \cos t$, we can show that

$$\lim_{n\to\infty} n\lambda_n(\mu_{\mathbb{T}}, e^{i\pi/2}) = 2\pi \frac{A-B}{\log A - \log B}$$

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With T_N, we can transform μ_σ back to a measure supported on the unit circle. Comparing it to $v(e^{it})dt$ yields

$$\lim_{n\to\infty} n\lambda_n(\mu_\sigma, z_0) = \frac{N}{|T_N'(z)|}\frac{A-B}{\log A - \log B} = \frac{1}{\omega_\sigma(z)}\frac{A-B}{\log A - \log B}.$$

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- Show that in a small neighbourhood of z_0, the integrals of the minimizing polynomial with respect to s_γ and s_σ are close to each other.

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- Show that in a small neighbourhood of z_0, the integrals of the minimizing polynomial with respect to s_γ and s_σ are close to each other.
- Use fast decreasing polynomials to show that $\lambda_n(\mu, z_0)$ can be estimated in terms of $\lambda_n(\mu_\sigma, z_0)$.

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- σ touches γ only at P,
- σ contains γ in its interior except for the point P,
- and if L denotes the enclosed lemniscate domain, then

$$\omega_\gamma(z_0) \leq \omega_\sigma(z_0) + \varepsilon.$$

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Thank you for your attention!